

A Framework on Moment Model Reduction for Kinetic Equation

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Abstract

By a further investigation on the structure of the coefficient matrix of the globally hyperbolic regularized moment equations for Boltzmann equation in [1], we propose a uniform framework to carry out model reduction to general kinetic equations, to achieve certain moment system. With this framework, the underlying reason why the globally hyperbolic regularization in [1] works is revealed. The even fascinating point is, with only routine calculation, existing models are represented and brand new models are discovered. Even if the study is restricted in the scope of the classical Grad's 13-moment system, new model with global hyperbolicity can be deduced.

1 Introduction

In 1949, the moment method was proposed by Grad [9] for the gas kinetic theory, and Grad's 13-moment theory is the most basic one among the moment models beyond the Navier-Stokes theory. It has been discovered in [14] that the 1D reduction of Grad's 13-moment equations is only hyperbolic around the equilibrium, and in [6], it is found that in 3D case, the equilibrium is on the boundary of the hyperbolicity region. Due to such a deficiency, the application of the moment method is strongly limited. During a long time, this remains a great obstacle to the improvement of the moment method. However, through the effort of a large number of researchers, the situation has become much more encouraging. Levermore investigated the maximum entropy method and showed in [12] that the moment system obtained by such a method owns global hyperbolicity. Unfortunately, it is difficult to put it into application owing to the lack of a simple analytical expression. Based on Levermore's 14-moment closure, an affordable 14-moment closure is proposed in [13] as an approximation, which also possesses almost global hyperbolicity. Meanwhile, the discovery of globally hyperbolic moment systems with large numbers of moments is also in progress. In 1D case, two types of globally hyperbolic moment systems are introduced in [1] and [11] using different strategies, and the approach in [11] successfully brought us some insight on the globally hyperbolic regularization in [1]. The method in [1] is extended to the multidimensional case in two different versions [2, 8], and some numerical results of the version in [2] are already carried out in [3].

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The research in this paper starts with a detailed study on the globally hyperbolic regularization of Grad's moment method proposed in [1]. The regularization introduced in [1] appears to be the result of a direct demand of hyperbolicity with specified characteristic speeds, and its underlying mechanism is not clarified, which makes its extension to the multidimensional case mysterious. By comparing the 1D regularized system with an adaptive discrete velocity model, the coefficient matrix in the regularized system is properly factorized and it is figured out how the system approximates the Boltzmann equation. This new finding spontaneously leads to a new deduction of the globally hyperbolic moment system, in which the time derivative and the spatial derivative are discretized in the same way so that one can achieve the global hyperbolicity by simply choosing a real diagonalizable operator to discrete the multiplication of the particle velocity. Due to this new deduction, the 1D globally hyperbolic moment system is instantly obtained, and may no longer be regarded as a regularization of the corresponding Grad's moment system.

A significant advantage of this new deduction is its extensibility. The whole procedure can be extended to the multidimensional Boltzmann with more flexible ansatz on the distribution function by only carrying out routine calculations. Based on some simple assumptions, a framework for deriving "moment systems" is established and we propose two conditions which ensure the hyperbolicity. By further exploration, we point out that these two conditions to hyperbolicity is hardly to be violated. This is on the very contrary of the studies on the moment models in extensive literatures, where the hyperbolicity is a subtle subject to be achieved. Using this framework, both multidimensional models in [2, 8] are reasonably deduced and a new globally hyperbolic 13-moment system is discovered. Furthermore, this new strategy can even be extended to a type of generic kinetic equations. As an example, the radiative transfer equation is investigated and it is found that the classic M_N model is also covered by the framework.

The rest of this paper is as follows: in Section 2, the Boltzmann equation and the moment method are briefly reviewed; in Section 3, the structure of the 1D globally hyperbolic moment equations is studied and a new deduction is proposed; in Section 4, the deduction is generalized to the multidimensional Boltzmann equation and later to the generic kinetic equation; finally, some concluding remarks are made in Section 5.

2 Moment Method for Boltzmann Equation

In the gas kinetic theory, the movement of the gas molecules is depicted from a statistical point of view. To be precise, it is supposed that at any specific time t and spatial point \mathbf{x} , the velocity of the gas molecule $\boldsymbol{\xi}$ is a random variable with probability density function $p[t, \mathbf{x}](\boldsymbol{\xi})$, and the corresponding distribution function is denoted as

$$f(t, \mathbf{x}, \boldsymbol{\xi}) = n(t, \mathbf{x}) \cdot p[t, \mathbf{x}](\boldsymbol{\xi}), \quad t \in \mathbb{R}^+, \quad \mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^D, \quad (2.1)$$

where $n(t, \mathbf{x})$ is the number density of the gas molecules, and D is the number of dimensions, which equals to 3 in our physical world. Denoting $\langle \phi \rangle = \int_{\mathbb{R}^D} \phi d\boldsymbol{\xi}$ and assuming the gas to be an ideal gas, we have

$$\rho(t, \mathbf{x}) = m_g \langle f \rangle, \quad \rho \mathbf{u}(t, \mathbf{x}) = m_g \langle \boldsymbol{\xi} f \rangle, \quad \rho(t, \mathbf{x}) RT(t, \mathbf{x}) = \frac{1}{D} m_g \langle |\boldsymbol{\xi}|^2 f \rangle, \quad (2.2)$$

where the macroscopic functions ρ , \mathbf{u} and T denotes the density, velocity and temperature respectively, and the constant R and m_g denotes the gas constant and the mass of a single molecule. By convention, we let $\theta(t, \mathbf{x}) = RT(t, \mathbf{x})$.

The distribution function is governed by the Boltzmann equation, which reads

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\mathbf{x}} f = \mathcal{S}(f), \quad (2.3)$$

where the right hand side $\mathcal{S}(f)$ is to model the effects of the interaction among particles. In this paper, the concrete form of $\mathcal{S}(f)$ is not concerned. We only assume here that if for some t_0 and \mathbf{x}_0 ,

$$f(t_0, \mathbf{x}_0, \boldsymbol{\xi}) = \exp(A + \mathbf{B} \cdot \boldsymbol{\xi} + C|\boldsymbol{\xi}|^2), \quad A \in \mathbb{R}, \quad \mathbf{B} \in \mathbb{R}^D, \quad C \in \mathbb{R}^-, \quad (2.4)$$

then $\mathcal{S}(f)(t_0, \mathbf{x}_0, \boldsymbol{\xi}) = 0$ (see e.g. [7]). Using the relations in (2.2), we can write the coefficients in (2.4) as expressions of ρ , \mathbf{u} and θ , and thus (2.4) is reformulated by

$$f(t_0, \mathbf{x}_0, \boldsymbol{\xi}) = \frac{\rho(t_0, \mathbf{x}_0)}{m[2\pi\theta(t_0, \mathbf{x}_0)]^{D/2}} \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{u}(t_0, \mathbf{x}_0)|^2}{2\theta(t_0, \mathbf{x}_0)}\right). \quad (2.5)$$

This distribution function is actually the local equilibrium of the gas.

By the assumption on the operator \mathcal{S} , the unknown function f has to be close to the local equilibrium if \mathcal{S} is large. Based on this observation, Grad [9] expanded the distribution into the Hermite series. Here we follow [4] and write the expansion as

$$f(t, \mathbf{x}, \boldsymbol{\xi}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^D} f_{\boldsymbol{\alpha}}(t, \mathbf{x}) \mathcal{H}_{\boldsymbol{\alpha}}^{[\theta(t, \mathbf{x})]} \left(\frac{\boldsymbol{\xi} - \mathbf{u}(t, \mathbf{x})}{\sqrt{\theta(t, \mathbf{x})}} \right). \quad (2.6)$$

In the above expansion, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_D)$ is a D -dimensional multi-index, and the basis functions $\mathcal{H}_{\boldsymbol{\alpha}}^{[\cdot]}$ is defined by

$$\mathcal{H}_{\boldsymbol{\alpha}}^{[\theta]}(\mathbf{v}) = m_g^{-1} \prod_{d=1}^D \frac{1}{\sqrt{2\pi}} \theta^{-\frac{\alpha_d+1}{2}} He_{\alpha_d}(v_d) \exp\left(-\frac{v_d^2}{2}\right), \quad \forall \mathbf{v} = (v_1, \dots, v_D) \in \mathbb{R}^D, \quad (2.7)$$

where $He_k(\cdot)$ is the Hermite polynomial of degree k , defined by

$$He_k(x) = (-1)^k \exp\left(\frac{x^2}{2}\right) \frac{d^k}{dx^k} \exp\left(-\frac{x^2}{2}\right). \quad (2.8)$$

It is obvious that if $f_{\boldsymbol{\alpha}} = 0$ for all $|\boldsymbol{\alpha}| \geq 1$, then f degenerates to the local equilibrium. Additionally, by substituting (2.6) into (2.2), one has

$$f_0 = \rho, \quad f_{\boldsymbol{\alpha}} \equiv 0 \quad \text{if } |\boldsymbol{\alpha}| = 1, \quad \sum_{|\boldsymbol{\alpha}|=1} f_{2\boldsymbol{\alpha}} \equiv 0. \quad (2.9)$$

Subsequently, the expansion (2.6) is substituted into the Boltzmann equation and the equations of $f_{\boldsymbol{\alpha}}$ are deduced by letting the coefficients of the expansion vanished. For simplicity, here we only consider a model case $D = 1$, where $\boldsymbol{\alpha}$ and \mathbf{u} are turned into scalars and we note them instead by α and u . The resulted equations are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0, \\ \rho \frac{\partial u}{\partial t} + \theta \frac{\partial \rho}{\partial x} + \rho u \frac{\partial u}{\partial x} + \rho \frac{\partial \theta}{\partial x} &= 0, \\ \frac{1}{2} \rho \frac{\partial \theta}{\partial t} + \frac{1}{2} \rho u \frac{\partial \theta}{\partial x} + \rho u \frac{\partial u}{\partial x} + 3 \frac{\partial f_3}{\partial x} &= 0, \\ \frac{\partial f_{\alpha}}{\partial t} - f_{\alpha-1} \frac{\theta}{\rho} \frac{\partial \rho}{\partial x} + (\alpha+1) f_{\alpha} \frac{\partial u}{\partial x} + \left(\frac{1}{2} \theta f_{\alpha-3} + \frac{\alpha-1}{2} f_{\alpha-1} \right) \frac{\partial \theta}{\partial x} \\ &\quad - \frac{3}{\rho} f_{\alpha-2} \frac{\partial f_3}{\partial x} + \theta \frac{\partial f_{\alpha-1}}{\partial x} + u \frac{\partial f_{\alpha}}{\partial x} + (\alpha+1) \frac{\partial f_{\alpha+1}}{\partial x} = \mathcal{S}_{\alpha}, \quad \alpha \geq 3, \end{aligned} \quad (2.10)$$

where \mathcal{S}_α is obtained by expansion of the collision term which is out of our concern in this paper, and we emphasize that $f_1 = f_2 = 0$ according to (2.9).

Obviously the equations in (2.10) form a system with infinite number of unknowns. In order to obtain a system with finite number of equations, Grad suggested to choose a positive integer $M \geq 2$, and then simply set f_α to be zero for all $\alpha > M$, and discard all the equations containing $\partial_t f_\alpha$ with $\alpha > M$. The resulting system is called Grad's $(M + 1)$ -moment system.

3 Globally Hyperbolic Regularized Moment System

In [1], the authors investigated the hyperbolicity¹ of 1D Grad's moment systems for any M , and concluded that for $M \geq 3$, Grad's moment system is only hyperbolic around the equilibrium. Moreover, a globally hyperbolic moment regularization to Grad's system was then proposed. However, it was not revealed in [1] the intrinsic structure of the regularized system, which makes the hyperbolicity achieved therein a mystery. In this section, the underlying structure of the globally hyperbolic regularization is discovered, and the hyperbolicity is clarified as a natural deduction.

3.1 Review of system in 1D case

We first review the globally hyperbolic moment system in [1]. We write Grad's $(M + 1)$ -moment system as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{S}(\mathbf{w}), \quad (3.2)$$

where $\mathbf{w} = (\rho, u, \theta, f_3, \dots, f_M)^T$, and the matrix $\mathbf{A}(\mathbf{w})$ is given by

$$\begin{pmatrix} u & \rho & 0 & \dots & 0 \\ \theta/\rho & u & 1 & 0 & \dots & 0 \\ 0 & 2\theta & u & 6/\rho & 0 & \dots & 0 \\ 0 & 4f_3 & \rho\theta/2 & u & 4 & 0 & \dots & 0 \\ -\theta f_3/\rho & 5f_4 & 3f_3/2 & \theta & u & 5 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\theta f_{M-2}/\rho & M f_{M-1} & \frac{1}{2}[(M-2)f_{M-2} + \theta f_{M-4}] & -3f_{M-3}/\rho & 0 & \dots & 0 & \theta & u & M \\ -\theta f_{M-1}/\rho & (M+1)f_M & \frac{1}{2}[(M-1)f_{M-1} + \theta f_{M-3}] & -3f_{M-2}/\rho & 0 & \dots & 0 & \theta & u \end{pmatrix}. \quad (3.3)$$

In [1], it is shown that this matrix is real diagonalizable only when $|f_M/(\rho\theta^{M/2})|$ and $|f_{M-1}/(\rho\theta^{(M-1)/2})|$ are small enough. If we modify the matrix $\mathbf{A}(\mathbf{w})$ to a "regularized coefficient matrix" $\hat{\mathbf{A}}(\mathbf{w})$ defined by

$$\hat{\mathbf{A}}(\mathbf{w}) := \mathbf{A}(\mathbf{w}) - (M+1)f_M \mathbf{E}_{M+1,2} - \frac{M+1}{2} f_{M-1} \mathbf{E}_{M+1,3}, \quad (3.4)$$

¹A first-order quasi-linear partial differential system

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{k=1}^D \mathbf{A}_k(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} = \mathbf{S}(\mathbf{w}) \quad (3.1)$$

is hyperbolic in some region Ω if and only if any linear combinations of $\mathbf{A}_k(\mathbf{w})$, $k = 1, \dots, D$ is real diagonalizable for all $\mathbf{w} \in \Omega$.

then the “regularized system”

$$\frac{\partial \mathbf{w}}{\partial t} + \hat{\mathbf{A}}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{S}(\mathbf{w}) \quad (3.5)$$

is globally hyperbolic. In equation (3.4), $\mathbf{E}_{p,q}$ stands for the $(M+1) \times (M+1)$ matrix (E_{ij}) with $E_{ij} = \delta_{ip}\delta_{jq}$. The characteristic polynomial of $\hat{\mathbf{A}}(\mathbf{w})$ is

$$\det[\lambda \mathbf{I} - \hat{\mathbf{A}}(\mathbf{w})] = \theta^{\frac{M+1}{2}} He_{M+1} \left(\frac{\lambda - u}{\sqrt{\theta}} \right). \quad (3.6)$$

Therefore the characteristic speeds of (3.5) are

$$u + c_j \sqrt{\theta}, \quad j = 0, \dots, M, \quad (3.7)$$

where c_0, \dots, c_M are the distinct roots of the polynomial $He_{M+1}(x)$.

3.2 Structure of the coefficient matrix

It has been discussed in [1] that the system (3.5) is similar as a “shifted and scaled discrete velocity model” whose discrete velocities are the characteristic speeds of the system. Our investigation starts from this point. A canonical discrete velocity model with characteristic speeds ξ_0, \dots, ξ_M has the following form:

$$\frac{\partial f_k^{\text{dvm}}}{\partial t} + \xi_k \frac{\partial f_k^{\text{dvm}}}{\partial x} = S_k^{\text{dvm}}, \quad k = 0, \dots, M. \quad (3.8)$$

Here $f_k^{\text{dvm}}(t, x)$ approximates $f(t, x, \xi_k)$, and $S_k^{\text{dvm}}(t, x)$ approximates $S(f)(t, x, \xi_k)$. For convenience, we write (3.8) as

$$\frac{\partial \mathbf{f}^{\text{dvm}}}{\partial t} + \mathbf{\Xi} \frac{\partial \mathbf{f}^{\text{dvm}}}{\partial x} = \mathbf{S}^{\text{dvm}}, \quad (3.9)$$

where

$$\mathbf{f}^{\text{dvm}} = (f_0^{\text{dvm}}, \dots, f_M^{\text{dvm}})^T, \quad \mathbf{S}^{\text{dvm}} = (S_0^{\text{dvm}}, \dots, S_M^{\text{dvm}})^T, \quad \mathbf{\Xi} = \text{diag}\{\xi_0, \dots, \xi_M\}. \quad (3.10)$$

In order to compare the regularized moment system (3.5) with (3.9), we first factorize the matrix $\hat{\mathbf{A}}(\mathbf{w})$ by

$$\hat{\mathbf{A}}(\mathbf{w}) = [\mathbf{L}(\mathbf{w})]^{-1} \mathbf{\Lambda}(\mathbf{w}) \mathbf{L}(\mathbf{w}), \quad \mathbf{\Lambda}(\mathbf{w}) = \text{diag} \left\{ u + c_0 \sqrt{\theta}, \dots, u + c_M \sqrt{\theta} \right\}, \quad (3.11)$$

and then substitute it into the system (3.5) and multiply both sides by $\mathbf{L}(\mathbf{w})$. This results in

$$\mathbf{L}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \mathbf{\Lambda}(\mathbf{w}) \mathbf{L}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{L}(\mathbf{w}) \mathbf{S}(\mathbf{w}). \quad (3.12)$$

Thus it is clear that $\mathbf{L}(\mathbf{w}) \partial_t \mathbf{w}$ corresponds to $\partial_t \mathbf{f}^{\text{dvm}}$, and $\mathbf{L}(\mathbf{w}) \partial_x \mathbf{w}$ corresponds to $\partial_x \mathbf{f}^{\text{dvm}}$. It can be verified for $M = 2$ that it is impossible to find a vector function $\mathbf{g}(\mathbf{w})$ such that $\partial_{t,x} \mathbf{g}(\mathbf{w}) = \mathbf{L}(\mathbf{w}) \partial_{t,x} \mathbf{w}$. Therefore, there does not exist a perfect correspondence between the regularized moment equations and the discrete velocity model. However, in the discrete velocity model, we may regard $\partial_{t,x} \mathbf{f}^{\text{dvm}}$ as the approximations of

$$\left(\frac{\partial f}{\partial(t, x)}(t, x, \xi_0), \dots, \frac{\partial f}{\partial(t, x)}(t, x, \xi_M) \right)^T.$$

Accordingly, $\mathbf{L}(\mathbf{w})\partial_{t,x}\mathbf{w}$ is regarded as the approximations of

$$\left(\frac{\partial f}{\partial(t,x)}(t,x,u+c_0\sqrt{\theta}), \dots, \frac{\partial f}{\partial(t,x)}(t,x,u+c_M\sqrt{\theta}) \right)^T.$$

Since $u+c_j\sqrt{\theta}$, $j=0, \dots, M$, are Hermite-Gauss quadrature points, such approximation is equivalent to approximating the functions $\partial_{t,x}f$ by a finite Hermite expansion

$$\frac{\partial f}{\partial(t,x)} \approx \sum_{\alpha=0}^M D_{\alpha}^{t,x}(\mathbf{w}) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi-u}{\sqrt{\theta}} \right), \quad (3.13)$$

where

$$\left(D_0^{t,x}(\mathbf{w}), \dots, D_M^{t,x}(\mathbf{w}) \right)^T = \mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w}) \partial_{t,x} \mathbf{w}, \quad (3.14)$$

and $\mathbf{T}^{[\theta]}$ is the transformation matrix from the values on the quadrature points to the coefficients. Now the system (3.12) can be rewritten as

$$\mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \left[\mathbf{T}^{[\theta]} \mathbf{\Lambda}(\mathbf{w}) \left(\mathbf{T}^{[\theta]} \right)^{-1} \right] \mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w}) \mathbf{S}(\mathbf{w}). \quad (3.15)$$

Below we are going to calculate the matrices $\mathbf{T}^{[\theta]} \mathbf{\Lambda}(\mathbf{w}) \left(\mathbf{T}^{[\theta]} \right)^{-1}$ and $\mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w})$.

Let $\mathbf{h} = (h_0, \dots, h_M)^T$ be a $(M+1)$ -dimensional vector and define

$$H(v) = \sum_{\alpha=0}^M h_{\alpha} \mathcal{H}_{\alpha}^{[\theta]}(v). \quad (3.16)$$

According to the definition of $\mathbf{T}^{[\theta]}$, we have

$$\left(\mathbf{T}^{[\theta]} \right)^{-1} \mathbf{h} = (H(c_0), \dots, H(c_M))^T, \quad (3.17)$$

and then

$$\mathbf{\Lambda}(\mathbf{w}) \left(\mathbf{T}^{[\theta]} \right)^{-1} \mathbf{h} = \left((u+c_0\sqrt{\theta})H(c_0), \dots, (u+c_M\sqrt{\theta})H(c_M) \right)^T. \quad (3.18)$$

If we define $\tilde{\mathbf{h}} = (\tilde{h}_0, \dots, \tilde{h}_M)^T = \mathbf{T}^{[\theta]} \mathbf{\Lambda}(\mathbf{w}) \left(\mathbf{T}^{[\theta]} \right)^{-1} \mathbf{h}$, and

$$\tilde{H}(v) = \sum_{\alpha=0}^M \tilde{h}_{\alpha} \mathcal{H}_{\alpha}^{[\theta]}(v), \quad (3.19)$$

then $\tilde{\mathbf{h}}$ is the unique vector such that

$$\left(\tilde{H}(c_0), \dots, \tilde{H}(c_M) \right)^T = \left((u+c_0\sqrt{\theta})H(c_0), \dots, (u+c_M\sqrt{\theta})H(c_M) \right)^T. \quad (3.20)$$

By the recursion relation of the Hermite polynomials, it is obtained that

$$\begin{aligned} (u+v\sqrt{\theta})H(v) &= \sum_{\alpha=0}^M h_{\alpha} \left[\theta \mathcal{H}_{\alpha+1}^{[\theta]}(v) + u \mathcal{H}_{\alpha}^{[\theta]}(v) + \alpha \mathcal{H}_{\alpha-1}^{[\theta]}(v) \right] \\ &= \sum_{\alpha=0}^M [\theta h_{\alpha-1} + u h_{\alpha} + (\alpha+1) h_{\alpha+1}] \mathcal{H}_{\alpha}^{[\theta]}(v) + \theta h_M \mathcal{H}_{M+1}^{[\theta]}(v), \end{aligned} \quad (3.21)$$

where $\mathcal{H}_{-1}^{[\theta]}(v)$, h_{-1} and h_{M+1} are taken as zeros. Since $\mathcal{H}_{M+1}^{[\theta]}(c_j) = 0$, $j = 0, \dots, M$, we have

$$(u + c_j \sqrt{\theta})H(c_j) = \sum_{\alpha=0}^M [\theta h_{\alpha-1} + u h_{\alpha} + (\alpha + 1)h_{\alpha+1}] \mathcal{H}_{\alpha}^{[\theta]}(c_j), \quad j = 0, \dots, M. \quad (3.22)$$

Collecting (3.19), (3.20) and (3.22) together, one instantly figures out that

$$\tilde{h}_{\alpha} = \theta h_{\alpha-1} + u h_{\alpha} + (\alpha + 1)h_{\alpha+1}, \quad \alpha = 0, \dots, M. \quad (3.23)$$

Thus the matrix $\mathbf{T}^{[\theta]} \mathbf{\Lambda}(\mathbf{w}) (\mathbf{T}^{[\theta]})^{-1}$ is obviously a tridiagonal matrix. The diagonal entries are all u , the subdiagonal entries are all θ , and the superdiagonal entries are $1, 2, \dots, M$. This matrix is denoted as $\mathbf{M}(u, \theta)$ below. The analysis reveals that $\mathbf{M}(u, \theta)$ actually acts as an operator which multiply a function like (3.16) by ξ , and then drop out the term with the highest degree.

In order to calculate the matrix $\mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w})$, we first consider small values of M , and calculate it directly using some computer algebra system. Then, by simple induction, we find that the matrix $\mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w})$ has the following general form:

$$\mathbf{D}(\mathbf{w}) = \begin{pmatrix} \mathbf{D}_{11}(\rho) & \mathbf{0} \\ \mathbf{D}_{21}(\mathbf{w}) & \mathbf{I}_{M-3} \end{pmatrix}, \quad (3.24)$$

where $\mathbf{D}_{11} = \text{diag}\{1, \rho, \rho/2, 1\}$, and

$$\mathbf{D}_{21} = \begin{pmatrix} 0 & f_3 & 0 & 0 \\ 0 & f_4 & \frac{1}{2}f_3 & 0 \\ 0 & f_5 & \frac{1}{2}f_4 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & f_{M-1} & \frac{1}{2}f_{M-2} & 0 \end{pmatrix}. \quad (3.25)$$

One can validate $\mathbf{D}(\mathbf{w}) = \mathbf{T}^{[\theta]} \mathbf{L}(\mathbf{w})$ by verifying $\mathbf{D}(\mathbf{w}) \mathbf{A}(\mathbf{w}) = \mathbf{M}(u, \theta) \mathbf{D}(\mathbf{w})$ with direct calculation. And finally the regularized moment system (3.5) is rewritten as

$$\mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \mathbf{M}(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{D}(\mathbf{w}) \mathbf{S}(\mathbf{w}). \quad (3.26)$$

In order to finish the comparison between the regularized moment system and the discrete velocity model, we turn back to the approximation (3.13) and try to find out how the right hand side of (3.13) approximates the derivatives on the left hand side. For convenience, we consider only the spatial derivative. Taking spatial derivatives on both sides of (2.6) with $D = 1$, we have the following expansion of $\partial_x f$:

$$\frac{\partial f}{\partial x} = \sum_{\alpha=0}^{+\infty} \left(\frac{\partial f_{\alpha}}{\partial x} + \frac{\partial u}{\partial x} f_{\alpha-1} + \frac{1}{2} \frac{\partial \theta}{\partial x} f_{\alpha-2} \right) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u}{\sqrt{\theta}} \right), \quad (3.27)$$

where $f_{-1} = f_{-2} = 0$ (See [5] for the detailed procedure). We calculate the coefficients $D_{\alpha}^x(\mathbf{w})$ using (3.24) and (3.14), and find out that the right hand side of (3.13) is

$$\sum_{\alpha=0}^M D_{\alpha}^x(\mathbf{w}) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u}{\sqrt{\theta}} \right) = \sum_{\alpha=0}^M \left(\frac{\partial f_{\alpha}}{\partial x} + \frac{\partial u}{\partial x} f_{\alpha-1} + \frac{1}{2} \frac{\partial \theta}{\partial x} f_{\alpha-2} \right) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u}{\sqrt{\theta}} \right). \quad (3.28)$$

Now it is clear that the approximation (3.13) is actually a direct truncation in the Hermite expansion.

As a summary, through a comparison with the discrete velocity model, we figure out the underlying mechanism of the regularized moment system, which evidently makes the reasonability and credibility of this new model more pronounced. And consequentially, this motivates us to propose the approach below to deduce the globally hyperbolic moment system directly.

3.3 New deduction procedure

In [1], the system is obtained by first deriving Grad's moment system and then applying the regularization. Based on the discussion above, we now may deduce the regularized moment system directly from the Boltzmann equation. The procedure is as follows:

1. Expand the distribution function into the Hermite series:

$$f(t, x, \xi) = \sum_{\alpha=0}^{+\infty} f_{\alpha}(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u}{\sqrt{\theta}} \right). \quad (3.29)$$

2. Calculate the time and spatial derivatives and the collision term:

$$\frac{\partial f}{\partial t} = \sum_{\alpha=0}^{+\infty} G_{\alpha}^t(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.30a)$$

$$\frac{\partial f}{\partial x} = \sum_{\alpha=0}^{+\infty} G_{\alpha}^x(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.30b)$$

$$S(f) = \sum_{\alpha=0}^{+\infty} Q_{\alpha}(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.30c)$$

where

$$G_{\alpha}^t = \frac{\partial f_{\alpha}}{\partial t} + \frac{\partial u}{\partial t} f_{\alpha-1} + \frac{1}{2} \frac{\partial \theta}{\partial t} f_{\alpha-2}, \quad G_{\alpha}^x = \frac{\partial f_{\alpha}}{\partial x} + \frac{\partial u}{\partial x} f_{\alpha-1} + \frac{1}{2} \frac{\partial \theta}{\partial x} f_{\alpha-2}, \quad (3.31)$$

and $Q_{\alpha}(t, x)$ depends on the collision model.

3. For the positive integer $M \geq 2$, apply a truncation on (3.30). The truncation drops off all f_{α} with $\alpha > M$, and discards all terms with $\alpha > M$ in the summations in (3.30). We write the results as

$$\frac{\partial f}{\partial t} \approx \sum_{\alpha=0}^M G_{\alpha}^t(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.32a)$$

$$\frac{\partial f}{\partial x} \approx \sum_{\alpha=0}^M G_{\alpha}^x(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.32b)$$

$$S(f) \approx \sum_{\alpha=0}^M \tilde{Q}_{\alpha}(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right). \quad (3.32c)$$

4. Calculate the convection term $\xi \partial_x f$ from (3.32b):

$$\xi \frac{\partial f}{\partial x} \approx \xi \sum_{\alpha=0}^M G_{\alpha}^x(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right) = \sum_{\alpha=0}^{M+1} J_{\alpha}(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.33)$$

where $J_{\alpha} = \theta G_{\alpha-1}^x + u G_{\alpha}^x + (\alpha + 1) G_{\alpha+1}^x$.

5. Truncate (3.33) again:

$$\xi \frac{\partial f}{\partial x} \approx \sum_{\alpha=0}^M J_{\alpha}(t, x) \mathcal{H}_{\alpha}^{[\theta]} \left(\frac{\xi - u(t, x)}{\sqrt{\theta(t, x)}} \right), \quad (3.34)$$

6. Substitute (3.32a), (3.32c) and (3.33) into the Boltzmann equation and extract the coefficients for the same basis functions. The result obtained

$$G_{\alpha}^t + J_{\alpha} = \tilde{Q}_{\alpha}, \quad 0 \leq \alpha \leq M \quad (3.35)$$

is the hyperbolic $(M + 1)$ -moment system.

Comparing with this procedure with the analysis in section 3.2, it is clear that the vectors $\mathbf{D}(\mathbf{w})\mathbf{w}_t$, $\mathbf{D}(\mathbf{w})\mathbf{w}_x$ and $\mathbf{D}(\mathbf{w})\mathbf{S}(\mathbf{w})$ are constructed by step 3, and the vector $\mathbf{M}(u, \theta)\mathbf{D}(\mathbf{w})\mathbf{w}_x$ is constructed by step 5.

By this procedure, the globally hyperbolic moment systems may be regarded directly as an approximation of Boltzmann equation, instead of the consequence of the regularization of Grad's moment systems. The key point during this deduction is that the truncation has to be applied instantly after every single operation, including both taking the derivatives of x or t , and the multiplication by velocity.

4 A Uniform Framework for Generic Kinetic Equation

In the last section, a brand new approach to deduce the globally hyperbolic moment system for Boltzmann equation is proposed. The approach is to be extended to a uniform framework hereafter. With this framework, the hyperbolicity of the obtained system can be intuitively perceived and no intricate manipulation of the coefficient matrices is needed. Based on the deduction for 1-dimensional problem in the last section, we first extend this to the multi-dimensional Boltzmann equation. And actually, this procedure can be made even more general. We will reveal this fact step by step in this section.

4.1 Multi-dimensional Boltzmann equation

Let us go back to the original Boltzmann equation (2.3). In order to include more general models, we consider a general expansion of the distribution function:

$$f(t, \mathbf{x}, \boldsymbol{\xi}) = \sum_{i=0}^{+\infty} F_i(t, \mathbf{x}) \varphi_i^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}). \quad (4.1)$$

Note that here we allow the basis functions φ_i to be dependent on some unknown vector function $\boldsymbol{\eta}(t, \mathbf{x}) = (\eta_1(t, \mathbf{x}), \dots, \eta_n(t, \mathbf{x}))$, and we suppose for any fixed $\boldsymbol{\eta}$, the basis functions $\{\varphi_i^{[\boldsymbol{\eta}]}(\boldsymbol{\xi})\}_{i=1}^{\infty}$ belong to some Hilbert space $\mathbb{H}^{[\boldsymbol{\eta}]}$ with real inner product $\langle \cdot, \cdot \rangle^{[\boldsymbol{\eta}]}$. Due to the parameter $\boldsymbol{\eta}$ in the basis function, some constraints

$$r_j(\eta_1, \dots, \eta_n, F_0, F_1, \dots, F_m) = 0, \quad j = 1, \dots, n \quad (4.2)$$

have to be imposed on the coefficients, and (4.2) is a non-degenerate algebraic system, which is provided by that its Jacobian matrix

$$\begin{pmatrix} \frac{\partial r_1}{\partial \eta_1} & \cdots & \frac{\partial r_1}{\partial \eta_n} & \frac{\partial r_1}{\partial F_0} & \cdots & \frac{\partial r_1}{\partial F_m} \\ \frac{\partial r_2}{\partial \eta_1} & \cdots & \frac{\partial r_2}{\partial \eta_n} & \frac{\partial r_2}{\partial F_0} & \cdots & \frac{\partial r_2}{\partial F_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r_n}{\partial \eta_1} & \cdots & \frac{\partial r_n}{\partial \eta_n} & \frac{\partial r_n}{\partial F_0} & \cdots & \frac{\partial r_n}{\partial F_m} \end{pmatrix},$$

has full row rank. For example, in expansion (3.29), we have $\boldsymbol{\eta} = (u, \theta)$, and the constraints are $f_1 = f_2 = 0$, $m = 2$.

We suppose the functions r_1, \dots, r_n are continuously differentiable. Owing to the non-degeneracy of (4.2) and the implicit function theorem, around any solution of (4.2), there exists one n -dimensional subvector of $(\eta_1, \dots, \eta_n, F_0, \dots, F_m)^T$ such that it can be written as a function of else variables. Thus, in order to construct a system with $(M+1)$ equations, we have to consider $(M+n+1)$ functions $\eta_1, \dots, \eta_n, F_0, F_1, \dots, F_M$, and then “close” the distribution function by writing all F_i , $i > M$ as algebraic functions of $\eta_1, \dots, \eta_n, F_0, F_1, \dots, F_M$. In order that (4.2) remains non-degenerate, we require $M \geq \max\{m, n\}$. Thus, the implicit function theorem can be applied to eliminate n unknowns, and we denote by \boldsymbol{w} the vector whose components are the remaining $(M+1)$ functions, and the distribution function f is actually approximated by

$$f(t, \boldsymbol{x}, \boldsymbol{\xi}) \approx \tilde{f}(t, \boldsymbol{x}, \boldsymbol{\xi}) = \sum_{i=0}^M F_i(t, \boldsymbol{x}) \varphi_i^{[\boldsymbol{\eta}]}(\boldsymbol{\xi}) + \sum_{i=M+1}^{+\infty} F_i(\boldsymbol{w}(t, \boldsymbol{x})) \varphi_i^{[\boldsymbol{\eta}]}(\boldsymbol{\xi}). \quad (4.3)$$

Now we mimic the procedure in section 3.3 to derive the equations for \boldsymbol{w} . We first calculate the time and spatial derivatives of \tilde{f} , together with the collision term:

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial(t, x_1, \dots, x_D)} &= \sum_{i=0}^{+\infty} [G_i^t(t, \boldsymbol{x}), G_i^{x_1}(t, \boldsymbol{x}), \dots, G_i^{x_D}(t, \boldsymbol{x})] \varphi_i^{[\boldsymbol{\eta}(t, \boldsymbol{x})]}(\boldsymbol{\xi}), \\ \mathcal{S}(\tilde{f}) &= \sum_{i=0}^{+\infty} Q_i(t, \boldsymbol{x}) \varphi_i^{[\boldsymbol{\eta}(t, \boldsymbol{x})]}(\boldsymbol{\xi}), \end{aligned} \quad (4.4)$$

where G_i^t and G_i^x are formulated as expressions of $\eta_1, \dots, \eta_n, F_0, F_1, \dots, F_M$ and their derivatives. In order to define the “truncation”, we denote by $\mathbb{S}_M^{[\boldsymbol{\eta}]}$ the linear subspace of $\mathbb{H}^{[\boldsymbol{\eta}]}$ spanned by $\{\varphi_0^{[\boldsymbol{\eta}]}, \dots, \varphi_M^{[\boldsymbol{\eta}]}\}$, and then approximate (4.4) by projecting it into this finite dimensional space $\mathbb{S}_M^{[\boldsymbol{\eta}]}$:

$$\frac{\partial \tilde{f}}{\partial(t, x_1, \dots, x_D)} \approx \sum_{i=0}^M [\tilde{G}_i^t(t, \boldsymbol{x}), \tilde{G}_i^{x_1}(t, \boldsymbol{x}), \dots, \tilde{G}_i^{x_D}(t, \boldsymbol{x})] \varphi_i^{[\boldsymbol{\eta}(t, \boldsymbol{x})]}(\boldsymbol{\xi}), \quad (4.5a)$$

$$\mathcal{S}(\tilde{f}) \approx \sum_{i=0}^M \tilde{Q}_i(t, \boldsymbol{x}) \varphi_i^{[\boldsymbol{\eta}(t, \boldsymbol{x})]}(\boldsymbol{\xi}). \quad (4.5b)$$

If $\varphi_0^{[\eta]}, \varphi_1^{[\eta]}, \dots$, are orthogonal to each other, then (4.5) is a direct truncation of (4.4), and otherwise, the coefficients in (4.5) has the following form:

$$\tilde{G}_i^s(t, \mathbf{x}) = \sum_{j=0}^{+\infty} a_{ij}(\boldsymbol{\eta}(t, \mathbf{x})) G_j^s(t, \mathbf{x}), \quad \tilde{Q}_i(t, \mathbf{x}) = \sum_{j=0}^{+\infty} a_{ij}(\boldsymbol{\eta}(t, \mathbf{x})) Q_j(t, \mathbf{x}), \quad (4.6)$$

$$s = t, x_1, \dots, x_D, \quad i = 0, \dots, M,$$

Here a_{ij} are only algebraic functions of $\boldsymbol{\eta}$, and they do not contain any derivatives of $\boldsymbol{\eta}$. Let $\tilde{\mathbf{g}}^s = (\tilde{G}_0^s, \dots, \tilde{G}_M^s)^T$, $s = t, x_1, \dots, x_D$. Using the chain rule and denoting all the unknown functions by \mathbf{w} , the vector $\tilde{\mathbf{g}}^s$ has the form

$$\tilde{\mathbf{g}}^s = \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial s}, \quad s = t, x_1, \dots, x_D. \quad (4.7)$$

Based on (4.5a), we approximate $\xi_k \partial_{x_k} f$ by

$$\xi_k \frac{\partial \tilde{f}}{\partial x_k} \approx \sum_{i=0}^{+\infty} J_{k,i}(t, \mathbf{x}) \varphi_i^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}), \quad k = 1, \dots, D, \quad (4.8)$$

and then project it into the space $\mathbb{S}_M^{[\eta]}$ similar as (4.5):

$$\xi_k \frac{\partial \tilde{f}}{\partial x_k} \approx \sum_{i=0}^M \tilde{J}_{k,i}(t, \mathbf{x}) \varphi_i^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}), \quad k = 1, \dots, D, \quad (4.9)$$

Finally, we put (4.9) and (4.5) into the Boltzmann equation, and make the following system by vanishing the coefficients of terms in the expansion:

$$\tilde{G}_i^t(t, \mathbf{x}) + \sum_{k=1}^D \tilde{J}_{k,i}(t, \mathbf{x}) = \tilde{Q}_i(t, \mathbf{x}), \quad i = 0, \dots, M. \quad (4.10)$$

Similar as (4.7), we let $\mathbf{l}_k = (\tilde{J}_{k,0}, \dots, \tilde{J}_{k,M})^T$ for $k = 1, \dots, D$, and then \mathbf{l}_k has the form of

$$\mathbf{l}_k = \mathbf{M}_k(\mathbf{w}) \tilde{\mathbf{g}}^{x_k}, \quad k = 1, \dots, D. \quad (4.11)$$

Thus, the system (4.10) is formulated as

$$\mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \sum_{k=1}^D \mathbf{M}_k(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} = \mathbf{q}(\mathbf{w}), \quad (4.12)$$

where $\mathbf{q} = (\tilde{Q}_0, \dots, \tilde{Q}_M)^T$. It is easy to find that the system (4.12) is hyperbolic if

1. $\mathbf{D}(\mathbf{w})$ is invertible;
2. any linear combination of $\mathbf{M}_k(\mathbf{w})$ is real diagonalizable.

We will point out later that these two conditions are fulfilled in very extensive configurations.

In the above procedure, the vector of unknown functions \mathbf{w} is defined only locally, as the result of the implicit function theorem. However, in a lot of cases (e.g. the case in section 3), \mathbf{w} can be defined in the large, or even globally. Thus the system (4.12) becomes globally hyperbolic only if \mathbf{w} is properly chosen that $\mathbf{D}(\mathbf{w})$ is invertible. Below, we give two examples to show how this abstract procedure works.

4.1.1 Example 1: 13-moment system

When the moment method was proposed by Grad [9], he immediately deduced a 13-moment system as its first application. However, this system suffers a serious problem with its hyperbolicity. It has been revealed in [6] that the hyperbolicity of this system cannot be ensured even around the equilibrium. To our current knowledge, there has no a 13-moment system with global hyperbolicity yet. Here we are going to deduce a globally hyperbolic 13-moment system using the framework introduced above.

In and only in this part, we will adopt the Einstein summation convention to simplify the notations. Since 13-moment system is based on 3D spatial and velocity spaces, the subscripts run over from 1 to 3. Following Grad's work, we first approximate the distribution function as

$$f(t, \mathbf{x}, \boldsymbol{\xi}) \approx f_{|13}(t, \mathbf{x}, \boldsymbol{\xi}) = \rho(t, \mathbf{x}) w^{[\theta(t, \mathbf{x})]}(\mathbf{C}(t, \mathbf{x})) + \kappa_i(t, \mathbf{x}) \mathcal{H}_i^{[\theta(t, \mathbf{x})]}(\mathbf{C}(t, \mathbf{x})) \\ + \frac{1}{2} \rho(t, \mathbf{x}) \theta_{\langle ij \rangle}(t, \mathbf{x}) \mathcal{H}_{ij}^{[\theta(t, \mathbf{x})]}(\mathbf{C}(t, \mathbf{x})) + \frac{1}{5} q_j(t, \mathbf{x}) \mathcal{H}_{ijj}^{[\theta(t, \mathbf{x})]}(\mathbf{C}(t, \mathbf{x})), \quad (4.13)$$

where θ_{ij} is a symmetric tensor, and

$$\theta_{\langle ij \rangle} = \theta_{ij} - \delta_{ij} \theta_{kk} / 3, \quad \mathbf{C} = \boldsymbol{\xi} - \mathbf{u}. \quad (4.14)$$

The basis functions are defined as

$$w^{[\theta]}(\mathbf{C}) = \frac{1}{m_g (2\pi\theta)^{3/2}} \exp\left(-\frac{\mathbf{C}_k \mathbf{C}_k}{2\theta}\right), \quad \mathcal{H}_{i_1 \dots i_n}^{[\theta]}(\mathbf{C}) = (-1)^n \frac{\partial^n w^{[\theta]}(\mathbf{C})}{\partial C_{i_1} \dots \partial C_{i_n}}. \quad (4.15)$$

In this case, $\boldsymbol{\eta} = (u_1, u_2, u_3, \theta)$, and the constraints (4.2) are

$$\kappa_1(t, \mathbf{x}) = \kappa_2(t, \mathbf{x}) = \kappa_3(t, \mathbf{x}) = 0, \quad \theta_{kk}(t, \mathbf{x}) = 3\theta(t, \mathbf{x}). \quad (4.16)$$

The inner product is defined by

$$\langle g_1, g_2 \rangle^{[\eta]} = \int_{\mathbb{R}^D} \frac{g_1(\boldsymbol{\xi}) g_2(\boldsymbol{\xi})}{w^{[\theta]}(\boldsymbol{\xi} - \mathbf{u})} d\boldsymbol{\xi}. \quad (4.17)$$

The vector \mathbf{w} of unknowns is chosen as

$$\mathbf{w} = (\rho, u_1, u_2, u_3, \theta_{11}, \theta_{22}, \theta_{33}, \theta_{12}, \theta_{13}, \theta_{23}, q_1, q_2, q_3)^T. \quad (4.18)$$

Taking derivatives on both sides of (4.13), we have

$$\frac{\partial f_{|13}}{\partial s} = \frac{\partial \rho}{\partial s} w^{[\theta]}(\mathbf{C}) + \rho \frac{\partial u_i}{\partial s} \mathcal{H}_i^{[\theta]}(\mathbf{C}) + \frac{1}{2} \rho \frac{\partial \theta}{\partial s} \mathcal{H}_{jj}^{[\theta]}(\mathbf{C}) + \frac{1}{2} \rho \theta_{\langle ij \rangle} \frac{\partial u_k}{\partial s} \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}) \\ + \frac{1}{5} \frac{\partial q_j}{\partial s} \mathcal{H}_{ijj}^{[\theta]}(\mathbf{C}) + \frac{1}{4} \rho \theta_{\langle ij \rangle} \frac{\partial \theta}{\partial s} \mathcal{H}_{ijkk}^{[\theta]}(\mathbf{C}) + \frac{1}{5} q_j \frac{\partial u_k}{\partial s} \mathcal{H}_{ijjk}^{[\theta]}(\mathbf{C}) + \frac{1}{10} q_j \frac{\partial \theta}{\partial s} \mathcal{H}_{ijjkk}^{[\theta]}(\mathbf{C}), \quad (4.19)$$

for $s = t, x_1, x_2, x_3$. Let \mathbf{B} be the vector whose components are the following 13 basis functions:

$$\mathbf{B} = \left(w^{[\theta]}(\mathbf{C}), \right. \\ \mathcal{H}_1^{[\theta]}(\mathbf{C}), \mathcal{H}_2^{[\theta]}(\mathbf{C}), \mathcal{H}_3^{[\theta]}(\mathbf{C}), \\ \mathcal{H}_{11}^{[\theta]}(\mathbf{C}), \mathcal{H}_{22}^{[\theta]}(\mathbf{C}), \mathcal{H}_{33}^{[\theta]}(\mathbf{C}), \mathcal{H}_{12}^{[\theta]}(\mathbf{C}), \mathcal{H}_{13}^{[\theta]}(\mathbf{C}), \mathcal{H}_{23}^{[\theta]}(\mathbf{C}), \\ \left. \mathcal{H}_{ii1}^{[\theta]}(\mathbf{C}), \mathcal{H}_{ii2}^{[\theta]}(\mathbf{C}), \mathcal{H}_{ii3}^{[\theta]}(\mathbf{C}) \right)^T, \quad (4.20)$$

and denote by $\mathbb{S}_{13}^{[\eta]}$ the linear space spanned by these basis functions. Then we can approximate (4.19) by projecting it into the 13-dimensional space $\mathbb{S}_{13}^{[\eta]}$. The result is

$$\begin{aligned} \frac{\partial f_{13}}{\partial s} &\approx \frac{\partial \rho}{\partial s} w^{[\theta]}(\mathbf{C}) + \rho \frac{\partial u_i}{\partial s} \mathcal{H}_i^{[\theta]}(\mathbf{C}) \\ &+ \frac{1}{2} \left(\rho \frac{\partial \theta_{ij}}{\partial s} + \frac{\partial \rho}{\partial s} \theta_{\langle ij \rangle} \right) \mathcal{H}_{ij}^{[\theta]}(\mathbf{C}) + \frac{1}{5} \left(\rho \theta_{\langle jk \rangle} \frac{\partial u_k}{\partial s} + \frac{\partial q_j}{\partial s} \right) \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}). \end{aligned} \quad (4.21)$$

The above approximation has the form

$$\frac{\partial f_{13}}{\partial s} \approx \mathbf{B}^T \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial s}, \quad (4.22)$$

where $\mathbf{D}(\mathbf{w})$ is the very matrix in (4.7). It is found from (4.21) that $\mathbf{D}(\mathbf{w})$ is a lower triangular square matrix with its diagonal entries as

$$1, \rho, \rho, \rho, \frac{1}{2}\rho, \frac{1}{2}\rho, \frac{1}{2}\rho, \rho, \rho, \rho, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}.$$

Therefore $\mathbf{D}(\mathbf{w})$ is invertible.

Now we calculate the convection term $\xi_k \partial_{x_k} f$ from (4.21):

$$\begin{aligned} \xi_k \frac{\partial f_{13}}{\partial x_k} &\approx \left(u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} \right) w^{[\theta]}(\mathbf{C}) + \left(\rho u_k \frac{\partial u_i}{\partial x_k} + \rho \frac{\partial \theta_{ik}}{\partial x_k} + \theta_{ik} \frac{\partial \rho}{\partial x_k} \right) \mathcal{H}_i^{[\theta]}(\mathbf{C}) \\ &+ \left(\frac{1}{2} \rho u_k \frac{\partial \theta_{ij}}{\partial x_k} + \frac{1}{2} u_k \theta_{\langle ij \rangle} \frac{\partial \rho}{\partial x_k} + \rho \theta \frac{\partial u_i}{\partial x_j} \right. \\ &\quad \left. + \frac{2}{5} \rho \theta_{\langle ik \rangle} \frac{\partial u_k}{\partial x_j} + \frac{1}{5} \rho \theta_{\langle kl \rangle} \delta_{ij} \frac{\partial u_k}{\partial x_l} + \frac{2}{5} \frac{\partial q_i}{\partial x_j} + \frac{1}{5} \delta_{ij} \frac{\partial q_k}{\partial x_k} \right) \mathcal{H}_{ij}^{[\theta]}(\mathbf{C}) \\ &+ \frac{1}{5} u_k \left(\rho \theta_{\langle jl \rangle} \frac{\partial u_l}{\partial x_k} + \frac{\partial q_j}{\partial x_k} \right) \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}) + \frac{1}{2} \theta \left(\rho \frac{\partial \theta_{ij}}{\partial x_k} + \theta_{\langle ij \rangle} \frac{\partial \rho}{\partial x_k} \right) \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}) \\ &+ \frac{1}{5} \theta \left(\rho \theta_{\langle jl \rangle} \frac{\partial u_l}{\partial x_k} + \frac{\partial q_j}{\partial x_k} \right) \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}). \end{aligned} \quad (4.23)$$

Projecting (4.23) into $\mathbb{S}_{13}^{[\eta]}$, one obtains

$$\begin{aligned} \xi_k \frac{\partial f_{13}}{\partial x_k} &\approx \left(u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} \right) w^{[\theta]}(\mathbf{C}) + \left(\rho u_k \frac{\partial u_i}{\partial x_k} + \rho \frac{\partial \theta_{ik}}{\partial x_k} + \theta_{ik} \frac{\partial \rho}{\partial x_k} \right) \mathcal{H}_i^{[\theta]}(\mathbf{C}) \\ &+ \left(\frac{1}{2} \rho u_k \frac{\partial \theta_{ij}}{\partial x_k} + \frac{1}{2} u_k \theta_{\langle ij \rangle} \frac{\partial \rho}{\partial x_k} + \rho \theta \frac{\partial u_i}{\partial x_j} \right. \\ &\quad \left. + \frac{2}{5} \rho \theta_{\langle ik \rangle} \frac{\partial u_k}{\partial x_j} + \frac{1}{5} \rho \theta_{\langle kl \rangle} \delta_{ij} \frac{\partial u_k}{\partial x_l} + \frac{2}{5} \frac{\partial q_i}{\partial x_j} + \frac{1}{5} \delta_{ij} \frac{\partial q_k}{\partial x_k} \right) \mathcal{H}_{ij}^{[\theta]}(\mathbf{C}) \\ &+ \frac{1}{5} \left(u_k \rho \theta_{\langle jl \rangle} \frac{\partial u_l}{\partial x_k} + u_k \frac{\partial q_j}{\partial x_k} + \rho \theta \frac{\partial \theta_{jk}}{\partial x_k} + \frac{1}{2} \rho \theta \frac{\partial \theta_{kk}}{\partial x_j} + \theta \theta_{\langle jk \rangle} \frac{\partial \rho}{\partial x_k} \right) \mathcal{H}_{ijk}^{[\theta]}(\mathbf{C}). \end{aligned} \quad (4.24)$$

Just like (4.11), the above approximation can be rewritten as

$$\xi_k \frac{\partial f_{13}}{\partial x_k} \approx \mathbf{B}^T \mathbf{M}_k(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k}, \quad (4.25)$$

where $\mathbf{M}_k(\mathbf{w})$, $k = 1, 2, 3$ are matrices with size 13×13 . Here we only give the expression of \mathbf{M}_1 :

$$\mathbf{M}_1(\mathbf{w}) = \begin{pmatrix} u_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta & u_1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \theta & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \theta & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3\theta}{5} & \frac{\theta}{5} & \frac{\theta}{5} & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\theta}{5} & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\theta}{5} & 0 & 0 & 0 & u_1 \end{pmatrix}, \quad (4.26)$$

and its real diagonalizability can be directly verified². Then the rotational invariance of the 13-moment expansion yields the real diagonalizability of any linear combination of $\mathbf{M}_k(\mathbf{w})$, $k = 1, 2, 3$. For Maxwell molecules, by expansion and projection, we have the following approximation of the collision term:

$$\mathcal{S}(f) \approx \mathbf{B}^T \mathbf{q}(\mathbf{w}) = -\frac{3\rho}{m_g} \chi^{(2,3)} \left(\frac{1}{2} \rho \theta_{\langle ij \rangle} \mathcal{H}_{ij}^{[\theta]}(\mathbf{C}) + \frac{2}{15} q_j \mathcal{H}_{ij}^{[\theta]}(\mathbf{C}) \right), \quad (4.27)$$

where $\chi^{(2,3)}$ is a constant and we refer the readers to [15] for its definition. Finally we have the following globally hyperbolic 13-moment system:

$$\frac{\partial \mathbf{w}}{\partial t} + [\mathbf{D}(\mathbf{w})]^{-1} \mathbf{M}_k(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} = [\mathbf{D}(\mathbf{w})]^{-1} \mathbf{q}(\mathbf{w}). \quad (4.28)$$

In explicit formation, the system is as

$$\begin{aligned} \frac{d\rho}{dt} + \rho \frac{\partial u_k}{\partial x_k} &= 0, \\ \frac{du_i}{dt} + \frac{\partial \theta_{ik}}{\partial x_k} + \frac{\theta_{ik}}{\rho} \frac{\partial \rho}{\partial x_k} &= 0, \quad i = 1, 2, 3, \\ \frac{d\theta_{ij}}{dt} - \theta_{ij} \frac{\partial u_k}{\partial x_k} + \frac{3}{5} \theta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \delta_{ij} \frac{\partial u_k}{\partial x_k} \right) + \frac{2}{5} \left(\theta_{ik} \frac{\partial u_k}{\partial x_j} + \theta_{jk} \frac{\partial u_k}{\partial x_i} + \delta_{ij} \theta_{kl} \frac{\partial u_k}{\partial x_l} \right) \\ &\quad + \frac{2}{5\rho} \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} + \delta_{ij} \frac{\partial q_k}{\partial x_k} \right) = -\frac{3\rho}{m_g} \chi^{(2,3)} \theta_{\langle ij \rangle}, \quad i, j = 1, 2, 3, \\ \frac{dq_j}{dt} - \theta_{\langle ij \rangle} \theta_{\langle ik \rangle} \frac{\partial \rho}{\partial x_k} - \rho \theta_{ij} \frac{\partial \theta_{ik}}{\partial x_k} + 2\rho \theta \frac{\partial \theta_{ij}}{\partial x_i} + \frac{1}{2} \rho \theta \frac{\partial \theta_{kk}}{\partial x_j} &= -\frac{2\rho}{m_g} \chi^{(2,3)} q_j, \quad j = 1, 2, 3. \end{aligned} \quad (4.29)$$

²We verify the real diagonalizability by validating that $p(\mathbf{M}_1) = 0$ for $p(\lambda) = (\lambda - u_1)[5(\lambda - u_1)^2 - 7\theta] \cdot [5(\lambda - u_1)^4 - 26\theta(\lambda - u_1)^2 + 15\theta^2]$, which is the minimal polynomial of \mathbf{M}_1 . Apparently $p(\lambda)$ factors completely into distinct linear factor when $\theta > 0$.

4.1.2 Example 2: full moment theories

In this section, we are going to reproduce the work of [2] and [8], where two types of globally hyperbolic moment systems are derived. Here we adopt the following approximation to the distribution function:

$$f(t, \mathbf{x}, \boldsymbol{\xi}) \approx \tilde{f}(t, \mathbf{x}, \boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^D \\ |\boldsymbol{\alpha}| \leq M}} f_{\boldsymbol{\alpha}}(t, \mathbf{x}) \mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\Theta}]}(\mathbf{C}(t, \mathbf{x})), \quad (4.30)$$

where

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}^T = (\theta_{ij})_{D \times D}, \quad \mathbf{C} = \boldsymbol{\xi} - \mathbf{u}(t, \mathbf{x}). \quad (4.31)$$

The basis functions are defined as

$$w^{[\boldsymbol{\Theta}]}(\mathbf{C}) = \frac{1}{m_g \sqrt{\det(2\pi\boldsymbol{\Theta})}} \exp\left(-\frac{1}{2} \mathbf{C}^T \boldsymbol{\Theta}^{-1} \mathbf{C}\right), \quad \mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\Theta}]}(\mathbf{C}) = (-1)^{|\boldsymbol{\alpha}|} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{C}^{\boldsymbol{\alpha}}} w^{[\boldsymbol{\Theta}]}(\mathbf{C}). \quad (4.32)$$

Here $\boldsymbol{\eta} = (u_1, \dots, u_D, \theta_{11}, \dots, \theta_{1D}, \theta_{22}, \dots, \theta_{2D}, \dots, \theta_{DD})$, and thus $D(D+3)/2$ constraints are needed. These constraints will be discussed later. We first define the inner product $\langle \cdot, \cdot \rangle^{[\boldsymbol{\eta}]}$ as

$$\langle g_1, g_2 \rangle^{[\boldsymbol{\eta}]} = \int_{\mathbb{R}^D} \frac{g_1(\boldsymbol{\xi}) g_2(\boldsymbol{\xi})}{w^{[\boldsymbol{\Theta}]}(\boldsymbol{\xi} - \mathbf{u})} d\boldsymbol{\xi}. \quad (4.33)$$

Mimicing the abstract procedure, we calculate $\partial_{t,\mathbf{x}} f$ and perform the projection to space $\mathbb{S}_M^{[\boldsymbol{\eta}]} = \text{span}\{\mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\Theta}]}(\mathbf{C}) \mid \boldsymbol{\alpha} \in \mathbb{N}^D, |\boldsymbol{\alpha}| \leq M\}$. The result is

$$\frac{\partial \tilde{f}}{\partial s} \approx \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^D \\ |\boldsymbol{\alpha}| \leq M}} \left(\frac{\partial f_{\boldsymbol{\alpha}}}{\partial s} + \sum_{i=1}^D \frac{\partial u_i}{\partial s} f_{\boldsymbol{\alpha} - \mathbf{e}_i} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial s} f_{\boldsymbol{\alpha} - \mathbf{e}_i - \mathbf{e}_j} \right) \mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\Theta}]}(\mathbf{C}), \quad (4.34)$$

where $s = t, x_1, \dots, x_D$. Now we calculate the convection term and apply the projection again:

$$\begin{aligned} \xi_k \frac{\partial \tilde{f}}{\partial x_k} \approx & \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^D \\ |\boldsymbol{\alpha}| \leq M}} \left[u_k \left(\frac{\partial f_{\boldsymbol{\alpha}}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i - \mathbf{e}_j} \right) \right. \\ & + (1 - \delta_{|\boldsymbol{\alpha}|, M})(\alpha_k + 1) \left(\frac{\partial f_{\boldsymbol{\alpha} + \mathbf{e}_k}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i + \mathbf{e}_k} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i - \mathbf{e}_j + \mathbf{e}_k} \right) \\ & \left. + \sum_{l=1}^D \theta_{kl} \left(\frac{\partial f_{\boldsymbol{\alpha} - \mathbf{e}_l}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i - \mathbf{e}_l} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\boldsymbol{\alpha} - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_l} \right) \right] \mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\eta}]}(\mathbf{C}), \\ & k = 1, \dots, D. \end{aligned} \quad (4.35)$$

Supposing the projected right hand side has the following approximation:

$$\mathcal{S}(f) \approx \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}^D \\ |\boldsymbol{\alpha}| \leq M}} Q_{\boldsymbol{\alpha}}(t, \mathbf{x}) \mathcal{H}_{\boldsymbol{\alpha}}^{[\boldsymbol{\eta}]}(\mathbf{C}), \quad (4.36)$$

we then have the following moment system:

$$\begin{aligned}
& \frac{\partial f_{\alpha}}{\partial t} + \sum_{i=1}^D \frac{\partial u_i}{\partial t} f_{\alpha-e_i} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial t} f_{\alpha-e_i-e_j} \\
& + \sum_{k=1}^D \left[u_k \left(\frac{\partial f_{\alpha}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\alpha-e_i} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\alpha-e_i-e_j} \right) \right. \\
& + (1 - \delta_{|\alpha|,M})(\alpha_k + 1) \left(\frac{\partial f_{\alpha+e_k}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\alpha-e_i+e_k} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\alpha-e_i-e_j+e_k} \right) \\
& \left. + \sum_{l=1}^D \theta_{kl} \left(\frac{\partial f_{\alpha-e_l}}{\partial x_k} + \sum_{i=1}^D \frac{\partial u_i}{\partial x_k} f_{\alpha-e_i-e_l} + \frac{1}{2} \sum_{i,j=1}^D \frac{\partial \theta_{ij}}{\partial x_k} f_{\alpha-e_i-e_j-e_l} \right) \right] = Q_{\alpha}, \\
& \alpha \in \mathbb{N}^D, \quad |\alpha| \leq M.
\end{aligned} \tag{4.37}$$

Before we check the hyperbolicity of this moment system, we have to give the constraints (4.2). Below we consider two cases, which correspond to the systems in [2] and [8] respectively.

First case: classic Hermite expansion We require the parameter η and the coefficients to satisfy

$$\begin{aligned}
f_{e_j} &= 0, \quad j = 1, \dots, D, & \sum_{j=1}^D f_{2e_j} &= 0, \\
\theta_{11} &= \theta_{22} = \dots = \theta_{DD}, & \theta_{ij} &= 0 \quad \text{if } i \neq j.
\end{aligned} \tag{4.38}$$

Define \mathbf{w}_m to be the vector whose components are all f_{α} with $|\alpha| = m$, and let

$$g_{e_i+e_j} = \frac{1}{2} \delta_{ij} \theta_{ij} + f_{e_i+e_j}, \quad i, j = 1, \dots, D, \quad \mathbf{g} = (g_{\alpha})|_{|\alpha|=2}. \tag{4.39}$$

Then the unknown vector \mathbf{w} can be chosen as

$$\mathbf{w} = (f_0, u_1, \dots, u_D, \mathbf{g}, \mathbf{w}_3, \dots, \mathbf{w}_M)^T. \tag{4.40}$$

It is not difficult to observe from (4.34) that the matrix $\mathbf{D}(\mathbf{w})$ (see eq. (4.7)) is a triangular matrix with all its diagonal entries nonzero, and thus the first hyperbolicity condition (invertibility of \mathbf{D}) is fulfilled. The second hyperbolicity condition (real diagonalizability of linear combinations of \mathbf{M}_k) can actually be satisfied in a very general case, as will be discussed in section 4.2.1. We claim that in this case, the hyperbolic moment system is exactly what is proposed in [2], which can be verified by direct comparison of their explicit expressions.

Second case: generalized Hermite expansion We require the parameter η and the coefficients to satisfy

$$f_{e_j} = 0, \quad j = 1, \dots, D, \quad f_{\alpha} = 0 \quad \text{if } |\alpha| = 2, \tag{4.41}$$

and define $\mathbf{w} = (f_0, \boldsymbol{\eta}, \mathbf{w}_3, \dots, \mathbf{w}_M)^T$. In this case, the matrix $\mathbf{D}(\mathbf{w})$ is also a triangular matrix with all the diagonal entries nonzero, and thus is invertible. The real diagonalizability of the linear combinations of \mathbf{M}_k will also be discussed in section 4.2.1, and we only claim here that this condition is also fulfilled, which leads to the global hyperbolicity of (4.37). With the constraints (4.41), the above procedure reproduces the systems derived in [8].

4.2 Generic kinetic equation

Here now on we are trying to extend the framework above to even generic kinetic equation formulated as

$$\frac{\partial f}{\partial t} + \mathbf{v}(\boldsymbol{\xi}) \cdot \frac{\partial f}{\partial \mathbf{x}} = \mathcal{S}(f). \quad (4.42)$$

Such equation arises from such as Liouville equation of Hamiltonian system, radiative transport or relativistic kinetic theory. For example, let us consider a general Hamiltonian system with its Hamiltonian to be $\mathcal{H}(t, \mathbf{x}, \boldsymbol{\xi})$. The probability density function $f(t, \mathbf{x}, \boldsymbol{\xi})$ in the configuration space is governed by the Liouville equation as

$$\frac{\partial f}{\partial t} + \frac{\partial \mathcal{H}}{\partial \boldsymbol{\xi}} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \mathcal{H}}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \boldsymbol{\xi}} = 0, \quad (4.43)$$

where the ordinate acceleration term $\partial_{\mathbf{x}} \mathcal{H} \cdot \partial_{\boldsymbol{\xi}} f$ is corresponding to $\mathcal{S}(f)$ in (4.42). Often no derivatives respected to \mathbf{x} is involved in this term. The convective velocity $\mathbf{v}(\boldsymbol{\xi})$ in (4.42) is then specified as $\partial_{\boldsymbol{\xi}} \mathcal{H}$. For the generic kinetic equation (4.42), due to its similarity to the Boltzmann equation, the procedure in section 4.1 can be applied with only slightly revision. Below we first give an outline of the deduction of the hyperbolic system.

1. Expand the function f into an infinite series (4.1), and specify the constraints (4.2) on the coefficients and parameters.
2. For a sufficiently large M , use the constraints (4.2) to eliminate n items from $\eta_1, \dots, \eta_n, F_0, F_1, \dots, F_M$, and denote by \mathbf{w} the vector whose components are the remaining items.
3. Approximate f by writing all F_i with $i > M$ as functions of \mathbf{w} (4.3).
4. Expand the time and spatial derivatives together with the right hand side into the series (4.4).
5. Through the inner product $\langle \cdot, \cdot \rangle^{[\boldsymbol{\eta}]}$, project (4.4) onto the finite dimensional space $\mathbb{S}_M^{[\boldsymbol{\eta}]} = \{\varphi_0^{[\boldsymbol{\eta}]}, \dots, \varphi_M^{[\boldsymbol{\eta}]}\}$ (see (4.5)–(4.7)), which provides us the approximation

$$\frac{\partial \tilde{f}}{\partial s} \approx \left(\varphi_0^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}), \dots, \varphi_M^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}) \right) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial s}, \quad s = t, x_1, \dots, x_D,$$

and

$$\mathcal{S}(\tilde{f}) \approx \sum_{i=0}^M \tilde{Q}_i(t, \mathbf{x}) \varphi_i^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}).$$

6. Calculate the convection term $v_k(\boldsymbol{\xi}) \partial_{x_k} f$ based on (4.5a), and expand the result into series:

$$v_k(\boldsymbol{\xi}) \frac{\partial \tilde{f}}{\partial x_k} \approx \sum_{i=0}^{+\infty} J_{k,i}(t, \mathbf{x}) \varphi_i^{[\boldsymbol{\eta}(t, \mathbf{x})]}(\boldsymbol{\xi}), \quad k = 1, \dots, D. \quad (4.44)$$

7. Project (4.44) onto $\mathbb{S}_M^{[\eta]}$:

$$v_k(\boldsymbol{\xi}) \frac{\partial \tilde{f}}{\partial x_k} \approx \sum_{i=0}^M \tilde{J}_{k,i}(t, \mathbf{x}) \varphi_i^{[\eta(t, \mathbf{x})]}(\boldsymbol{\xi}), \quad k = 1, \dots, D. \quad (4.45)$$

Then by (4.11) we also have

$$\tilde{J}_{k,i} = \mathbf{M}_k(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k}, \quad k = 1, \dots, D.$$

8. Put (4.45) and (4.5a) into (4.42), and finally obtain the system as (4.10),

$$\mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial t} + \sum_{k=1}^D \mathbf{M}_k(\mathbf{w}) \mathbf{D}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} = \mathbf{q}(\mathbf{w}),$$

which is a reduced model for the generic kinetic equation (4.42), where $\mathbf{q} = (\tilde{Q}_0, \dots, \tilde{Q}_M)^T$.

Again, the matrices $\mathbf{D}(\mathbf{w})$ and $\mathbf{M}_k(\mathbf{w})$ appear at the same locations in the above deduction, and the system is hyperbolic if $\mathbf{D}(\mathbf{w})$ is invertible and any linear combination of $\mathbf{M}_k(\mathbf{w})$ is real diagonalizable.

Below, we will first look into these two conditions leading to hyperbolicity, and then provide the M_N model for radiative transport as the application of the framework.

4.2.1 On the conditions to hyperbolicity

In order to give the expressions of the matrices $\mathbf{D}(\mathbf{w})$ and $\mathbf{M}_k(\mathbf{w})$, we denote $\{\psi_0^{[\eta]}, \dots, \psi_M^{[\eta]}\}$ as an orthogonal basis of the $(M+1)$ -dimensional space $\mathbb{S}_M^{[\eta]}$, which satisfies

$$\langle \psi_i^{[\eta]}, \psi_j^{[\eta]} \rangle^{[\eta]} = \delta_{ij}, \quad i, j = 0, \dots, M. \quad (4.46)$$

Let $\mathbf{b}^{[\eta]} = (\varphi_0^{[\eta]}, \dots, \varphi_M^{[\eta]})^T$ and $\tilde{\mathbf{b}}^{[\eta]} = (\psi_0^{[\eta]}, \dots, \psi_M^{[\eta]})^T$. Clearly, the orthogonal basis $\tilde{\mathbf{b}}^{[\eta]}$ can be obtained by Schmidt's orthogonalization from $\mathbf{b}^{[\eta]}$. Then there exists a non-singular matrix $\mathbf{T}^{[\eta]}$ such that $\tilde{\mathbf{b}}^{[\eta]} = \mathbf{T}^{[\eta]} \mathbf{b}^{[\eta]}$ ³. According to (4.3), the function \tilde{f} can actually be written as a function of \mathbf{w} and $\boldsymbol{\xi}$, which is denoted by $\tilde{f}(t, \mathbf{x}, \boldsymbol{\xi}) = F(\mathbf{w}(t, \mathbf{x}); \boldsymbol{\xi})$. Thus, the matrices $\mathbf{D}(\mathbf{w})$ and $\mathbf{M}_k(\mathbf{w})$ can be written as

$$\mathbf{D} = \mathbf{T}^{[\eta]} \tilde{\mathbf{D}}, \quad \tilde{\mathbf{D}} = \left\langle \tilde{\mathbf{b}}^{[\eta]}, \left(\frac{\partial F}{\partial \mathbf{w}} \right)^T \right\rangle^{[\eta]}, \quad (4.47a)$$

$$\mathbf{M}_k = \mathbf{T}^{[\eta]} \tilde{\mathbf{M}}_k \left(\mathbf{T}^{[\eta]} \right)^{-1}, \quad \tilde{\mathbf{M}}_k = \left\langle \tilde{\mathbf{b}}^{[\eta]}, v_k(\boldsymbol{\xi}) \left(\tilde{\mathbf{b}}^{[\eta]} \right)^T \right\rangle^{[\eta]}, \quad k = 1, \dots, D, \quad (4.47b)$$

where the notation $\langle \tilde{\mathbf{a}}, \mathbf{a}^T \rangle^{[\eta]}$ stands for the matrix with its entries as $\langle \tilde{a}_i, a_j \rangle^{[\eta]}$. Through (4.47b), we immediately get the following criterion on the real diagonalizability of \mathbf{M}_k :

Theorem 1. *Any linear combination of \mathbf{M}_k is real diagonalizable if the inner product $\langle \cdot, \cdot \rangle^{[\eta]}$ satisfies*

$$\langle v_k(\boldsymbol{\xi}) g_1, g_2 \rangle^{[\eta]} = \langle g_1, v_k(\boldsymbol{\xi}) g_2 \rangle^{[\eta]}, \quad k = 1, \dots, D \quad (4.48)$$

for any $g_1, g_2 \in \mathbb{S}_M^{[\eta]}$.

³Precisely, the matrix $\mathbf{T}^{[\eta]}$ can be a lower triangular matrix provided by Schmidt's orthogonalization.

Proof. Let a_k , $k = 1, \dots, D$ be D arbitrary reals. By equation (4.47b), we know that $\sum_k a_k \mathbf{M}_k$ is real diagonalizable if and only if $\sum_k a_k \tilde{\mathbf{M}}_k$ is real diagonalizable. According to (4.48),

$$\begin{aligned} \sum_{k=1}^D a_k \tilde{\mathbf{M}}_k^T &= \sum_{k=1}^D a_k \left\langle v_k(\boldsymbol{\xi}) \tilde{\mathbf{b}}^{[\eta]}, \left(\tilde{\mathbf{b}}^{[\eta]} \right)^T \right\rangle^{[\eta]} \\ &= \sum_{k=1}^D a_k \left\langle \tilde{\mathbf{b}}^{[\eta]}, v_k(\boldsymbol{\xi}) \left(\tilde{\mathbf{b}}^{[\eta]} \right)^T \right\rangle^{[\eta]} = \sum_{k=1}^D a_k \tilde{\mathbf{M}}_k, \end{aligned} \quad (4.49)$$

which means $\sum_k a_k \tilde{\mathbf{M}}_k$ is a symmetric matrix, and thus is real diagonalizable. \square

Obviously, both inner products (4.17) and (4.33) satisfy the hypothesis of the above theorem, and then the real diagonalizability of linear combinations of \mathbf{M}_k can be directly obtained for both cases. The condition (4.48) implies a quite natural symmetry on the inner product equipped by $\mathbb{S}_M^{[\eta]}$. Clearly, an inner product formulated as

$$\langle g_1, g_2 \rangle^{[\eta]} = \int g_1 g_2 w^{[\eta]}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

for any weight function $w^{[\eta]}(\boldsymbol{\xi})$, fulfils this condition, while this formation actually includes almost every case of practical interests. Thus the model derived by our framework is globally hyperbolic for most situations, only if the the unknowns is properly chosen that the matrix \mathbf{D} is invertible.

Now we consider the matrix \mathbf{D} . Under this framework, the unknown vector \mathbf{w} is adopted as the parameters to construct a distribution function $F(\mathbf{w}; \boldsymbol{\xi})$ in $\mathbb{S}_M^{[\eta]}$ to approximate the solution of the kinetic equation. Definitely, it is not permitted that a distribution function $F(\mathbf{w}; \boldsymbol{\xi})$ is represented by two different set of parameters \mathbf{w} . Otherwise, the uniqueness of the model is already destroyed at the very beginning — only due to the representation of the solution. Thus it is necessary that

$$\mathbf{w}^0 \neq \mathbf{w}^1 \implies F(\mathbf{w}^0; \boldsymbol{\xi}) \neq F(\mathbf{w}^1; \boldsymbol{\xi}). \quad (4.50)$$

At a given \mathbf{w}^0 , we may linearize $F(\mathbf{w}^1; \boldsymbol{\xi})$ around \mathbf{w}^0 to have

$$F(\mathbf{w}^1; \boldsymbol{\xi}) \approx F(\mathbf{w}^0; \boldsymbol{\xi}) + \frac{\partial F}{\partial \mathbf{w}}(\mathbf{w}^0; \boldsymbol{\xi})(\mathbf{w}^1 - \mathbf{w}^0). \quad (4.51)$$

By (4.47a) and (4.51), one notice that $\partial_{\mathbf{w}} F$ is an approximation of the multiplication of \mathbf{D} and the basis functions, by dropping off the terms out of the approximation space $\mathbb{S}_M^{[\eta]}$, which results in

$$F(\mathbf{w}^1; \boldsymbol{\xi}) \approx F(\mathbf{w}^0; \boldsymbol{\xi}) + \left(\mathbf{b}^{[\eta]}(\boldsymbol{\xi}) \right)^T \mathbf{D}(\mathbf{w}^0)(\mathbf{w}^1 - \mathbf{w}^0).$$

By (4.50), it is natural to require the matrix \mathbf{D} to be invertible.

Remark 1. However, sometimes the matrix $\mathbf{D}(\mathbf{w})$ may be singular on some isolated points while the mapping from \mathbf{w} to the projection of $F(\mathbf{w}; \boldsymbol{\xi})$ onto $\mathbb{S}_M^{[\eta]}$ remains injective. For example, we consider the case $D = 1$ and only one basis function

$$F(\eta, \xi) = \varphi^{[\eta]}(\xi) = \exp(-\xi^2 + \eta^3). \quad (4.52)$$

Let $w = \eta$, and then $\partial_w F = 3\eta^2 F$. In this case, $\partial_w F = 0$ when $w = 0$, but the mapping $w \mapsto F(\eta, \xi)$ is an injection. Such singularity is often caused by the existence of saddle points in the mapping, and this can be removed by using alternative \mathbf{w} locally. In (4.52), if we choose $w = \eta^3$, then $\partial_w F$ becomes nonzero around $\eta = 0$.

4.2.2 Example 3: M_N model for radiative transport

Consider the radiative transfer equation

$$\frac{1}{c} \frac{\partial f}{\partial t} + \mathbf{v}(\boldsymbol{\xi}) \cdot \frac{\partial f}{\partial \mathbf{x}} = \mathcal{C}(f; T), \quad (4.53)$$

where c is the speed of light, the right hand side \mathcal{C} models interactions between photons and the background medium with T as the material temperature, and

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3)^T, & \boldsymbol{\xi} &= (\theta, \phi)^T \in [0, \pi] \times [0, 2\pi), \\ \mathbf{v}(\boldsymbol{\xi}) &= (v_1(\boldsymbol{\xi}), v_2(\boldsymbol{\xi}), v_3(\boldsymbol{\xi}))^T = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T. \end{aligned} \quad (4.54)$$

For any integer $N \geq 1$, define

$$\mathcal{M}^{[\eta]} = \left[\exp \left(-\frac{\hbar \nu c}{k_B} \sum_{m=0}^N \sum_{l=-m}^m \eta_{lm} Y_{lm}(\boldsymbol{\xi}) \right) - 1 \right]^{-1}, \quad (4.55)$$

where $Y_{lm}(\boldsymbol{\xi})$ are real spherical harmonics, and η_{lm} are the corresponding coefficients. The parameters \hbar , ν and k_B denote Planck's constant, the frequency and Boltzmann's constant, respectively. Then we define basis functions as

$$\varphi_{lm}^{[\eta]}(\boldsymbol{\xi}) = Y_{lm}(\boldsymbol{\xi}) \left(1 + \mathcal{M}^{[\eta]} \right) \mathcal{M}^{[\eta]}, \quad l = -m, \dots, m, \quad (4.56)$$

and the Hilbert spaces $\mathbb{H}^{[\eta]}$ and $\mathbb{S}_N^{[\eta]}$:

$$\mathbb{H}^{[\eta]} = \text{span} \left\{ \varphi_{lm}^{[\eta]} \mid m = 0, 1, \dots, \quad l = -m, \dots, m \right\}, \quad (4.57a)$$

$$\mathbb{S}_N^{[\eta]} = \text{span} \left\{ \varphi_{lm}^{[\eta]} \mid m = 0, 1, \dots, N, \quad l = -m, \dots, m \right\}, \quad (4.57b)$$

$$\langle g_1, g_2 \rangle^{[\eta]} = \int_0^{2\pi} \int_0^\pi g_1(\boldsymbol{\xi}) g_2(\boldsymbol{\xi}) \left[\left(1 + \mathcal{M}^{[\eta]} \right) \mathcal{M}^{[\eta]} \right]^{-1} d\theta d\phi. \quad (4.57c)$$

Using power series expansion, we have

$$\exp \left(\frac{\hbar \nu c}{k_B} \sum_{m=0}^N \sum_{l=-m}^m \eta_{lm} Y_{lm}(\boldsymbol{\xi}) \right) = \sum_{m=0}^{+\infty} \sum_{l=-m}^m a_{lm}^{[\eta]} Y_{lm}(\boldsymbol{\xi}). \quad (4.58)$$

Now we approximate the function f by

$$\begin{aligned} \tilde{f}(t, \mathbf{x}, \boldsymbol{\xi}) &= \frac{2\hbar\nu^3}{c^2} \mathcal{M}^{[\eta(t, \mathbf{x})]} = \frac{2\hbar\nu^3}{c^2} \left[1 - \exp \left(\frac{\hbar \nu c}{k_B} \sum_{m=0}^N \sum_{l=-m}^m \eta_{lm} Y_{lm}(\boldsymbol{\xi}) \right) \right] \left(1 + \mathcal{M}^{[\eta]} \right) \mathcal{M}^{[\eta]} \\ &= \frac{2\hbar\nu^3}{c^2} (1 - a_{00}^{[\eta(t, \mathbf{x})]}) \varphi_{00}^{[\eta(t, \mathbf{x})]}(\boldsymbol{\xi}) - \sum_{m=0}^{+\infty} \sum_{l=-m}^m \frac{2\hbar\nu^3}{c^2} a_{lm}^{[\eta(t, \mathbf{x})]} \varphi_{lm}^{[\eta(t, \mathbf{x})]}(\boldsymbol{\xi}) \in \mathbb{H}^{[\eta]}. \end{aligned} \quad (4.59)$$

We choose $\mathbf{w} = \boldsymbol{\eta}$ and the constraints are spontaneously provided by (4.59).

Taking time and spatial derivatives on both sides of (4.59), we have

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial s} &= \frac{2\hbar^2 \nu^4}{k_{\text{BC}}} \sum_{m=0}^N \sum_{l=-m}^m \frac{\partial \eta_{lm}}{\partial s} Y_{lm}(\boldsymbol{\xi}) \left(1 + \mathcal{M}^{[\boldsymbol{\eta}]}\right) \mathcal{M}^{[\boldsymbol{\eta}]} \\ &= \frac{2\hbar^2 \nu^4}{k_{\text{BC}}} \sum_{m=0}^N \sum_{l=-m}^m \frac{\partial \eta_{lm}}{\partial s} \varphi_{lm}^{[\boldsymbol{\eta}]}(\boldsymbol{\xi}), \quad s = t, x_1, x_2, x_3.\end{aligned}\tag{4.60}$$

which is already a function in $\mathbb{S}_N^{[\boldsymbol{\eta}]}$ so that no additional projection is needed. Meanwhile, it is clear that $\mathbf{D} = \frac{2\hbar^2 \nu^4}{k_{\text{BC}}} \mathbf{I}$ which is obviously invertible.

In order to obtain the hyperbolic system, one still needs to write the matrices $\mathbf{M}_k(\boldsymbol{\eta})$ and project $\mathcal{C}(\tilde{f}; T)$ onto $\mathbb{S}_N^{[\boldsymbol{\eta}]}$. All these calculations are routine and here we omit the details which are quite tedious. We comment that the resulting system is globally hyperbolic since the inner product (4.57c) satisfies the hypothesis of Theorem 1, and the initial approximation (4.59) reveals that this moment system coincides with the M_N model [10].

5 Concluding remarks

Through investigation on a special type of one-dimensional hyperbolic models, a general framework for the construction of hyperbolic models from the kinetic equations has been established. We have shown that this framework already covers a number of existing hyperbolic models (the 1D and n D hyperbolic regularizations of Grad's moment methods [1, 2], the globally hyperbolic moment systems with generalized Hermite expansion [8] and the M_N model in radiative transfer [10]), and can also be used to discover new hyperbolic models. Actually, some more models such as the classic discrete velocity model, the maximum entropy model [12] and the quadrature-based projection method [11] are also included.

For a first order model with Cauchy data, the hyperbolicity is necessary to the existence of the solution. Historically, the hyperbolicity of different models is a quite subtle problem and has to be analysed case by case with great patience, which eventually leads to some prejudices against the moment method. It seems that the new perception on the hyperbolicity of the moment equation in this paper may bring us the dawn of hope to further development of the moment method for kinetic equations.

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References

- [1] Z. Cai, Y. Fan, and R. Li. Globally hyperbolic regularization of Grad's moment system in one dimensional space. *Comm. Math Sci.*, 11(2):547–571, 2013.
- [2] Z. Cai, Y. Fan, and R. Li. Globally hyperbolic regularization of Grad's moment system. *Comm. Pure Appl. Math.*, 67(3):464–518, 2014.

- [3] Z. Cai, Y. Fan, R. Li, and Z. Qiao. Dimension-reduced hyperbolic moment method for the Boltzmann equation with BGK-type collision. To appear in *Commun. Comput. Phys.*
- [4] Z. Cai and R. Li. Numerical regularized moment method of arbitrary order for Boltzmann-BGK equation. *SIAM J. Sci. Comput.*, 32(5):2875–2907, 2010.
- [5] Z. Cai, R. Li, and Y. Wang. Numerical regularized moment method for high Mach number flow. *Commun. Comput. Phys.*, 11(5):1415–1438, 2012.
- [6] Z.-N. Cai, Y.-W. Fan, and R. Li. On hyperbolicity of 13-moment system. *arXiv:1401.7523*, 2013.
- [7] S. Chapman and T. G. Cowling. *The Mathematical Theory of Non-uniform Gases, Third Edition*. Cambridge University Press, 1990.
- [8] Y.-W. Fan and R. Li. Globally hyperbolic moment system by generalized Hermite expansion. *arXiv:1401.4639*, 2014.
- [9] H. Grad. On the kinetic theory of rarefied gases. *Comm. Pure Appl. Math.*, 2(4):331–407, 1949.
- [10] C. D. Hauck. High-order entropy-based closures for linear transport in slab geometry. *Commun. Math. Sci.*, 9(1):187–205, 2011.
- [11] J. Koellermeier, R. Schaerer, and M. Torrilhon. A framework for hyperbolic approximation of kinetic equations using quadrature-based projection methods. submitted.
- [12] C. D. Levermore. Moment closure hierarchies for kinetic theories. *J. Stat. Phys.*, 83(5–6):1021–1065, 1996.
- [13] J. McDonald and M. Torrilhon. Affordable robust moment closures for CFD based on the maximum-entropy hierarchy. *J. Comput. Phys.*, 251:500–523, 2013.
- [14] I. Müller and T. Ruggeri. *Rational Extended Thermodynamics, Second Edition*, volume 37 of *Springer tracts in natural philosophy*. Springer-Verlag, New York, 1998.
- [15] H. Struchtrup. *Macroscopic Transport Equations for Rarefied Gas Flows: Approximation Methods in Kinetic Theory*. Springer, 2005.